A COORDINATE SYSTEM FOR A NON-SPHERICAL EARTH - OBLATE SPHEROIDAL COORDINATES

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ABSTRACT. Oceanography texts note that the earth is not a perfect sphere, that geopotential surfaces are not adequately described using spherical coordinates and that oblate spheroidal coordinates are more appropriate. They usually go on to state that oblate spheroidal coordinates are based on an ellipse for the surface of the earth and hyperbolas for marking latitude but do not give details on how such a coordinate system is constructed from arbitrary ellipses and hyperbolas. In this document we provide some additional detail on how an oblate spheroidal coordinate is constructed and how it may be applied to the earth.

1. INTRODUCTION

Oceanography texts such as Gill [1] note that the earth is not a perfect sphere, that geopotential surfaces are not adequately described using spherical coordinates and that oblate spheroidal coordinates are more appropriate. They usually go on to state that oblate spheroidal coordinates are based on an ellipse for the surface of the earth and hyperbolas for marking latitude but do not give details on how such a coordinate system is constructed from arbitrary ellipses and hyperbolas.

In this document we provide some additional detail on how an oblate spheroidal coordinate is constructed and how it may be applied to the earth.

In Section 2 we describe orthogonal coordinate systems and why they are important.

In Section 3 we examine the first of two orthogonal coordinate systems, the spherical coordinate system, and how it may be applied to the earth in Section 4.

In Sections 5 and 6 we look at some general properties of ellipses and hyperbolas before examining in Section 7 the second of orthogonal coordinate systems, the oblate spheroidal coordinate system.

In Section 8 we show how an oblate spheroidal coordinate system may be applied to the earth.

2. ORTHOGONAL COORDINATE SYSTEMS

Information in this section is taken from Moon and Spencer [2].

An orthogonal coordinate system (u_1, u_2, u_3) may be designated by the *metric coefficients*, g_{11}, g_{22}, g_{33}. An infinitesimal distance is written

$$(ds)^2 = g_{11}(\mathrm{d}u_1)^2 + g_{22}(\mathrm{d}u_2)^2 + g_{33}(\mathrm{d}u_3)^2 \quad (1)$$

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where

$$g_{ii} = \left(\frac{\partial x_1}{\partial u_i}\right)^2 + \left(\frac{\partial x_2}{\partial u_i}\right)^2 + \left(\frac{\partial x_3}{\partial u_i}\right)^2 \tag{2}$$

and x_i are rectangular coordinates.

Basically this means that for a sufficiently small volume around any point the three axes of the coordinate system can be considered to be orthogonal. For two disjoint points the axes are not necessarily parallel, e.g. in a rectangular coordinate system the axes for disjoint points are parallel but they are not in a spherical coordinate system.

Knowing the metric coefficients, one can write the equations for volume, gradient, curt, etc. [1] shows that infinitesimal *distances* along the coordinate axes are $(g_{11})^{\frac{1}{2}}\mathrm{d}u_1$, $(g_{22})^{\frac{1}{2}}\mathrm{d}u_2$, $(g_{33})^{\frac{1}{2}}\mathrm{d}u_3$. Using these, an element of area on the u_1u_2 surface is

$$\mathrm{d}\mathcal{A} = [(g_{11})^{\frac{1}{2}}\mathrm{d}u_1][(g_{22})^{\frac{1}{2}}\mathrm{d}u_2] = (g_{11}g_{22})^{\frac{1}{2}}\mathrm{d}u_1\mathrm{d}u_2 \tag{3}$$

Similarly, an element of *volume* in $u_1u_2u_3$ is

$$\mathrm{d}\mathcal{V} = (g_{11}g_{22}g_{33})^{\frac{1}{2}}\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3. \tag{4}$$

Gradient in orthogonal curvilinear coordinates (u_1, u_2, u_3) is

$$\nabla\phi = \frac{\mathbf{a}_1}{(g_{11})^{\frac{1}{2}}}\frac{\partial\phi}{\partial u_1} + \frac{\mathbf{a}_2}{(g_{22})^{\frac{1}{2}}}\frac{\partial\phi}{\partial u_2} + \frac{\mathbf{a}_3}{(g_{33})^{\frac{1}{2}}}\frac{\partial\phi}{\partial u_3} \tag{5}$$

where $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ are unit vectors.

Similarly, operators such *divergence, curl* and *Laplacian* as well as the partial differential equations such as *Laplace's equation* and the *wave equations* may be expressed in orthogonal coordinates. See [2] for details.

Most importantly, the partial differential equations of classical physics such as the Laplace's equation, Poisson's equation, and the scalar and vector wave equations are separable when expressed in an orthogonal coordinate system, i.e. can be separated into three ordinary differential equations.

In [2] 40 coordinate systems are described in relation to rectangular coordinates, metric coefficients and expressions are given for operators such as divergence, gradient, curl.

We shall examine in this document spherical and oblate spheroidal coordinates. The equations for these coordinate systems will be expressed using a reference *rectangular coordinate system* with coordinates (x, y, z).

3. Spherical Coordinates

To construct a *spherical coordinate system* with respect a rectangular coordinate system place the *origin* of the system at $(0, 0, 0)$ in the rectangular system.

The positive z axis is *axis line.*

The x, y plane is the *equatorial plane* and angles are measured from the positive x axis in a counter-clockwise direction with respect to the z axis.

To determine the spherical coordinates, $(r,\ \theta,\ \phi)$, for a point (x, y, z) not coincident with the origin, draw a *radius line* from the origin. (See Figure 1.) The length of the radius line is the r coordinate and has as a value any non-negative number.

Measure the angle from the positive z axis to the radius line. This angle is the θ coordinate and has a value in the range $[0,\ \pi]$.

Project the radius line onto the x, y plane. Measure the angle between the projection and the positive x axis. This angle is the ϕ coordinate and has a value in the range $[0,\ 2\pi)$.

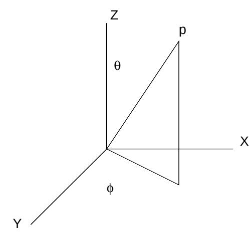

FIGURE 1. Determining Spherical Coordinates of a Point

Surfaces of constant r are spheres.

Surfaces of constant θ are concentric cones with the vertex of the cone at the origin and the axis of cone parallel to z axis.

Surfaces of constant ϕ are planes containing the axis line.

A point coincident with the origin has coordinates $(0,\ 0,\ 0)$. A point on the axis line at a distance d from the origin has coordinates $(d,\ 0,\ 0)$ if it is on the positive z axis and $(-d,\ 0,\ 0)$ otherwise.

Following the treatment in [2]

$$u_1 = r, \qquad 0 \leq r < \infty \tag{6}$$

$$u_2 = \theta, \qquad 0 \leq \theta \leq \pi \tag{7}$$

$$u_3 = \phi, \qquad 0 \leq \phi < 2\pi \tag{8}$$

$$x = r \sin\theta \cos\phi \tag{9}$$

$$y = r \sin\theta \sin\phi \tag{10}$$

$$z = r \cos\theta \tag{11}$$

with metric coefficients

$$g_{11} = 1 \qquad g_{22} = r^2 \qquad g_{33} = r^2 \sin^2\theta \tag{12}$$

4. Applying Spherical Coordinates to the Earth

The r coordinate is the distance from the center of the earth.

The θ coordinate is the angle measured from the north pole. To convert θ to latitude

$$0 \leq \theta \leq \frac{\pi}{2} \qquad \frac{\pi}{2} - \theta = \text{North latitude}$$

$$\frac{\pi}{2} \leq \theta \leq \pi \qquad \theta - \frac{\pi}{2} = \text{South latitude}$$

$$0 \leq \theta \leq 90^\circ \qquad 90^\circ - \theta = \text{North latitude}$$

$$90^\circ \leq \theta \leq 180^\circ \qquad \theta - 90^\circ = \text{South latitude}$$

For example, θ of 10° corresponds to 80° N latitude, 100° degrees to 10° S latitude, The ϕ coordinate is the eastward angle from the prime meridian. To convert ϕ to longitude

$0 \le \phi \le \pi$	$\phi =$ East longitude
$\pi \le \phi \le 2\pi$	$2\pi - \phi =$ West longitude

$0 \le \phi \le 180°$	$\phi =$ East longitude
$\pi \le \phi \le 360°$	$360° - \phi =$ West longitude

For example, ϕ of $10°$ corresponds to $10°$ E longitude, $190°$ degrees to $170°$ W longitude.

5. Representations for an Ellipse

One representation for an ellipse in the x, z plane with its *major axis* parallel to the x axis and its *minor axis* parallel to the z axis is (see Figure 2)

$$\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1 \tag{13}$$

where a is the major axis and b is the minor axis and $a < b$.

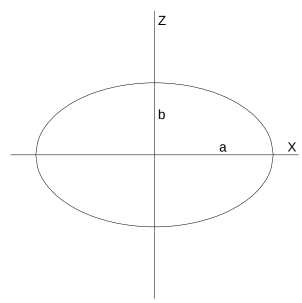

Figure 2. Ellipse w/ major & minor axes in x, z plane

Another representation is

$$\frac{x^2}{a^2} + \frac{z^2}{a^2(1-e^2)} = 1 \tag{14}$$

where a is the major axis and e is the *eccentricity*. e is

$$e = \sqrt{1 - \frac{b^2}{a^2}} \tag{15}$$

and since $b < a$ then $0 \le e < 1$.

Noting that $0 \le \tanh(\zeta) < 1$ for all ζ, another representation is

$$\frac{x^2}{a^2} + \frac{z^2}{a^2 \tanh^2 \zeta} = 1 \tag{16}$$

or equivalently

$$\frac{x^2}{c^2 \cosh^2 \eta} + \frac{z^2}{c^2 \sinh^2 \eta} = 1 \tag{17}$$

It is straightforward to determine the values for c and η in terms of the major and minor axes

$$\tanh(\eta) = \frac{b}{a} \tag{18}$$

$$c = \frac{a}{\cosh \eta} \tag{19}$$

The ellipses in Figure 4 have the same c but varying η. The ellipses in Figure 4

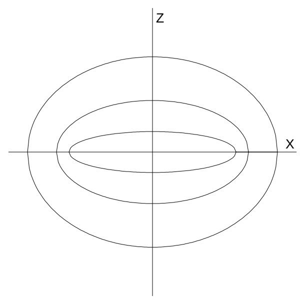

FIGURE 3. Ellipses: Constant c, Varying η

have varying c but the same η.

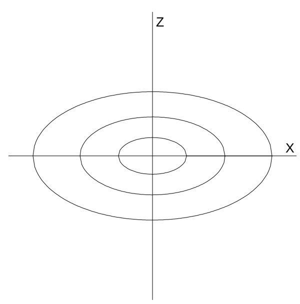

FIGURE 4. Ellipses: Varying c, Constant η

6. REPRESENTATIONS FOR A HYPERBOLA

A hyperbola with its axis parallel to the x axis, *asymptote* g/f and coordinate of the inflection points on the x axis at $\pm f$ can be represented as (see Figure 5)

$$\frac{x^2}{f^2} - \frac{z^2}{g^2} = 1 \tag{20}$$

Note that it is the ratio g/f that determines the slope of the asymptote but the value of f determines the hyperbola's inflection point. A family of curves can be drawn that have the same asymptote; the value of the inflection point selects a unique member from the family.

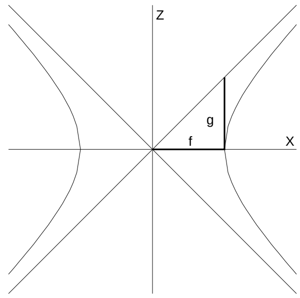

FIGURE 5. Hyperbola and Asymptote

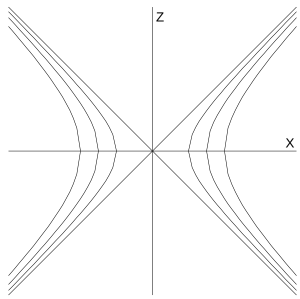

FIGURE 6. Hyperbolas: Same Asymptote, Varying Inflection Point

Another representation for a hyperbola is

$$\frac{x^2}{h^2 \sin^2 \theta} - \frac{z^2}{h^2 \cos^2 \theta} = 1 \tag{21}$$

where θ is measured from the positive z axis and may have any value in the range $[0, \pi]$. Note that θ and $(\pi - \theta)$ produce the same hyperbola, i.e. varying θ over the range $[0, \pi/2]$ is sufficient to generate all slope values.

For a given value of θ there is a whole family of curves with the same asymptote $(\pi/2 - \theta)$.

The inflection point, $h\,sin(\theta)$, selects a unique member from the family.

The ellipses in Figure 7 have the same h but varying θ. The ellipses in Figure 8

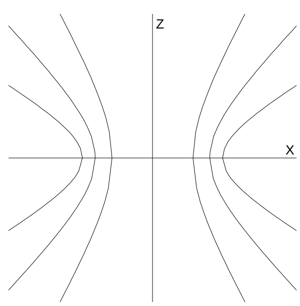

FIGURE 7. Hyperbolas: Constant h, Varying θ

have varying h but the same θ.

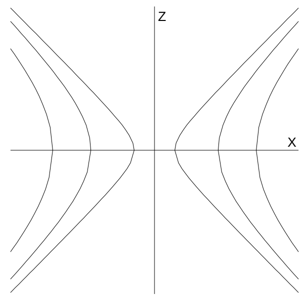

FIGURE 8. Hyperbolas: Varying h, Constant θ

7. Oblate Spheroidal Coordinates

An oblate spheroidal coordinate system, (r, θ, ϕ), has an origin and an axis of rotation coincident with the positive z axis. An ellipse in the x, z plane with major axis parallel to the x axis and minor axis parallel to the z axis is used to characterize an r coordinate. A hyperbola with its axis parallel to the x axis is used to characterize latitude θ. The third coordinate, ϕ, is measured the same as in spherical coordinates.

The key to creating an orthogonal coordinate system is connecting the equations for the ellipse and hyperbola. We use (17) to describe the ellipses and (21) to describe the hyperbolas.

In the x, z plane the equations are

$$\frac{x^2}{c^2 \cosh^2 \eta} + \frac{z^2}{c^2 \sinh^2(\eta)} = 1 \tag{22}$$

for the ellipse and

$$\frac{x^2}{c^2 \sin^2(\theta)} - \frac{z^2}{c^2 \cos^2(\theta)} = 1 \tag{23}$$

for hyperbola.

η in (22) corresponds to a radius and θ in (23) to the co-latitude.

For a given value of c we have one particular oblate spheroidal coordinate system of concentric ellipses and hyperbolas.

Given a c, a particular value of η generates a particular ellipse and a particular value of θ generates a particular hyperbola. For a given c, for all values of η and θ the intersection of the ellipses and hyperbolas are orthogonal.

If we now rotate the ellipse and hyperbola about the z-axis we have a three dimensional orthogonal coordinate system using coordinates (η, θ, ϕ).

Combining Figures 3 and 7 to get Figure 9 using the same value for c we get an oblate spheroidal coordinate system in the x, z plane.

Note that as η goes to zero in (22) the minor axis of the ellipse goes to zero and the major axis to c, *not to zero*. Note also that the inflection point for $\theta = \pi/2$ that corresponds to 0°N is also c.

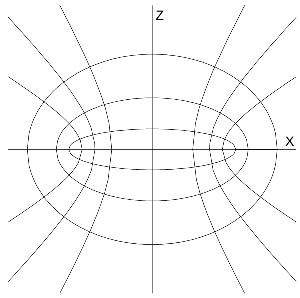

FIGURE 9. Oblate Spheroidal Coordinates: Constant c, Varying η, θ

Following the treatment in [2]

$$u_1 = \eta, \qquad 0 \leq \eta < \infty \tag{24}$$

$$u_2 = \theta, \qquad 0 \leq \theta \leq \pi \tag{25}$$

$$u_3 = \phi, \qquad 0 \leq \phi < 2\pi \tag{26}$$

$$x = c \cosh \eta \sin \theta \cos \phi \tag{27}$$

$$y = c \cosh \eta \sin \theta \sin \phi \tag{28}$$

$$z = c \sinh \eta \cos \theta \tag{29}$$

with metric coefficients

$$g_{11} = g_{22} = c^2(\cosh^2 \eta - \sin^2 \theta) \qquad g_{33} = c^2 \cosh^2 \eta \sin^2 \theta \tag{30}$$

8. Applying Oblate Spheroidal Coordinates to the Earth

Using (18) and (19) and values for the major axis and minor axes of the earth, 6378 km and 6356 km, we can determine c

$$\tanh(\eta) = \frac{6356}{6378} \tag{31}$$

$$\eta = 3.180 \tag{32}$$

$$c = \frac{6378}{\cosh(\eta)} \tag{33}$$

$$c = 529.29 \tag{34}$$

The resulting oblate spheroidal coordinate system that has been adjusted to the eccentricity of the Earth is

$$\frac{x^2}{529.29^2 \cosh^2(\eta)} + \frac{z^2}{529.29^2 \sinh^2(\eta)} = 1 \tag{35}$$

$$\frac{x^2}{529.29^2 \sin^2(\theta)} - \frac{z^2}{529.29^2 \cos^2(\theta)} = 1 \tag{36}$$

and with $\eta = 3.180$ in (35) the ellipse is the surface of the earth.

A particular value η creates an ellipse with a "radius" η. A particular value of θ intersects the ellipse and creates North and South co-latitude circles. ϕ maps to latitude as described for spherical coordinates.

References

[1] A.E. Gill, *Atmosphere - Ocean Dynamics,* Academic Press, 1982
[2] P. Moon, D.E. Spencer, *Field Theory Handbook,* Springer-Verlag, 1971.

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